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Wave radiation by a submerged source undergoing large amplitude periodic motion

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Abstract. Equations are derived to calculate the water waves radiation at infinity by a submerged source undergoing large amplitude motion. These equations do not require the full solution of the velocity potential itself, as demonstrated by a number of two- and three-dimensional examples. The results obtained are used to derive a far field equation for calculating the steady force (the drift force) on a submerged body undergoing large amplitude motion. It is concluded that the equations derived are useful to cases such as a deeply submerged body for which the source distribution may be taken as those obtained in an unbounded fluid domain.

1. Introduction

There exist many mathematical identities between various physical parameters in marine hydrodynamics. For the fully linearized velocity potential theory the most well known may be the Haskind relation [1], which relates the exciting force due to wave diffraction of an incoming wave to the radiated wave due to body oscillation. Other well known identities are that between wave reflection and wave transmission [2], and that between the radiated wave and diffracted wave [3]. These identities not only provide a better understanding of physical problem but also enable us to obtain some results without solving the corresponding problem.

For the second order velocity potential theory, various equations have been derived to obtain the desired results without the solution of the second order potential. Lighthill [4] and Molin [5] derived equations in the infinite and the finite water depth, respectively, which calculate the second order diffraction force on the body from the first order potential alone. Their equations were modified by Wu and Eatock Taylor [6] for the two-dimensional case, which has a contribution from infinity. Further equations were derived by Eatock Taylor et al. [7] to calculate the second order pressure on the body surface and Wu [8] to calculate the second order wave reflection and transmission by a two-dimensional horizontal cylinder.

A common feature in these derivations is that all of them use the Green's identity which relates to integration over a surface to that over the volume enclosed by this surface. Since the potential satisfies the Laplace equation in the fluid domain, it is then possible to relate the integration of two potentials (either physical or artificial) over the part of the surface to that over the rest of the surface. Evidently, such a principle can be applied to many other cases in marine hydrodynamics. Here we shall consider the problem of wave radiation by a submerged hydrodynamic source undergoing large amplitude motion. We shall take into account not only the variation of the source strength but also that of its position, provided these variations have the same period. We shall assume that the disturbance on the free surface remains small and linearization of the free surface boundary condition can still be adopted. Such a mathematical model has been used in several occasions [9–12]. Here we shall show that to obtain the radiated wave by a submerged source there is no need to obtain the potential itself following the principle discussed above. We shall consider three examples: (1) a two-dimensional source in open water, (2) a three-dimensional source in a channel, (3) a three-dimensional source in the open water. The results obtained are then used to derive far field equations for drift forces on a submerged body in these cases.

For most practical bodies, the source distribution is unknown. Its solution requires that of the corresponding Green function. This means the above results cannot be directly used. In many cases, however, the source distribution can be obtained by some approximate method, especially for a slender body or a deeply submerged body. It is known in the fully linearized problem that when far field equations are used the source distribution obtained in an unbounded fluid domain can be adopted for a deeply submerged body. The results are usually quite satisfactory. On the other hand, for a deeply submerged body undergoing large amplitude motion, the non-linearity due to the change its position may have to be taken into account but the disturbance on the free surface remains small and the linearized boundary condition still applies. Thus, in these cases, the equations obtained in this paper are particularly useful.

2. The two-dimensional source

We define a coordinate system $o-xz$ so that the origin is located on the undisturbed free surface and z points upwards. We consider the problem of a submerged source oscillating periodically with mean position at (x_0, z_0). Based on assumption of the ideal flow, the velocity potential ϕ satisfies the following governing equation and the boundary conditions

$$\nabla^2\phi = \sigma(t)\,\delta[x - x_0 + f_1(t)]\,\delta[z - z_0 + f_2(t)] \tag{1}$$

in the whole fluid domain R, where $\sigma(t)$ is the strength of the source and $\delta(x)$ is the Delta function;

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \tag{2}$$

on the free surface S_F or $z = 0$ and

$$\frac{\partial\phi}{\partial z} = 0 \tag{3}$$

on the bottom of the fluid S_B or $z = -d$. The radiation conditions at $x = \pm\infty$ require that waves propagate outwards. We assume that $\sigma(t)$, $f_1(t)$ and $f_2(t)$ in equation (1) are of the same period T and the potential has become a periodic function of time in the entire fluid domain. We shall not attempt to consider the transient motion, because unless time tends to infinity the fluid at $x = \pm\infty$ is not disturbed and there is no wave there. For the periodic motion, we may write the asymptotical expansion of the potential at $x = \pm\infty$ as

$$\phi = \sum_{m=0}^{\infty} \frac{\cosh[k_m(z+d)]}{\cosh k_m d}\,[A_m^{\pm}\cos(k_m x \mp m\omega t) + B_m^{\pm}\sin(k_m x \mp m\omega t)] + u_{\pm}x, \quad x \to \pm\infty \tag{4}$$

where

$$\omega = T/2\pi \tag{5a}$$

$$k_m \tanh(k_m d) = (m\omega)^2/g \tag{5b}$$

The velocity potential satisfying the above equations may be solved using the method for a submerged circular cylinder undergoing large amplitude motion [12]. Here we shall show that to obtain the coefficients in equation (4), there is no need to obtain the potential itself.

u_+ and u_- in equation (4) are related to the 'blockage' of the two-dimensional free surface flow with finite depth [13]. Using Gauss' theorem, we have

$$\int_S \frac{\partial \phi}{\partial n} \, \mathrm{d}S = \int_R \nabla^2 \phi \, \mathrm{d}R$$

where $\mathbf{n}$ is the normal of S pointing out of the fluid domain, $S = S_F + S_B + S_\infty$ and S_∞ comprises two vertical lines at $x = \pm\infty$. From equations (1), (2) and (3), we obtain

$$-\frac{1}{g}\int_{S_F} \phi_{tt} \, \mathrm{d}S + \int_{S_\infty} \frac{\partial \phi}{\partial n} \, \mathrm{d}R = \sigma(t)$$

integrating both sides of the equation with respect to t from 0 to T and using equations (4), we have

$$u_+ - u_- = \frac{1}{d}\int_0^T \sigma(t) \, \mathrm{d}t \, . \tag{6}$$

This shows that if there is no net flow from the source, $u_+ = u_-$.

For the other coefficients, we only consider $m > 0$ because only these terms will generate waves. We define

$$\psi_\pm = \cosh[k_m(z + d)] \sin(k_m x \pm m\omega t) \tag{7}$$

which satisfies the Laplace equation, free surface and bottom boundary conditions. Using Green's identity, we have

$$\int_S \left(\psi_\pm \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi_\pm}{\partial n}\right) \mathrm{d}S = \int_R \psi_\pm \nabla^2 \phi \, \mathrm{d}R \, .$$

From equation (1) and boundary conditions on ϕ and ψ, we obtain

$$-\frac{1}{g}\frac{\partial}{\partial t}\int_{S_F} \left(\frac{\partial \psi_\pm}{\partial t}\phi - \frac{\partial \phi}{\partial t}\psi_\pm\right) \mathrm{d}S + \int_{S_\infty} \left(\psi_\pm \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi_\pm}{\partial n}\right) \mathrm{d}S$$
$$= \sigma(t) \cosh\{k_m[d + z_0 - f_3(t)]\} \sin\{k_m[x_0 - f_1(t)] \pm m\omega t\} \, .$$

Integrating both sides with respect to t from 0 to T and using equations (4) and (7), we obtain

$$A_m^\pm = -\frac{4 \cosh k_m d}{T(2k_m d + \sinh 2k_m d)} \int_0^T \sigma(t) \cosh\{k_m[d + z_0 - f_3(t)]\} \sin\{k_m[x_0 - f_1(t)] \mp m\omega t\} \, \mathrm{d}t \, . \tag{8}$$

Similarly if we define

$$\psi_{\pm} = \cosh[k_m(z+d)]\cos(k_m x \pm m\omega t) \tag{9}$$

and follow the procedure above, we obtain

$$B_m^{\pm} = -\frac{4\cosh k_m d}{T(2k_m d + \sinh 2k_m d)} \int_0^T \sigma(t)\cosh\{k_m[d + z_0 - f_3(t)]\}\cos\{k_m[x_0 - f_1(t)] \mp m\omega t\}\,\mathrm{d}t\,. \tag{10}$$

3. A three dimensional source in a channel

We now consider the problem of a source in a channel with depth d, width b and infinite length. The coordinate system $o - xyz$ is defined so that x is in the longitudinal direction, z points upwards and origin is located on the free surface and the centre of the channel. The potential due to a source at (x_0, y_0, z_0) satisfies the following equations

$$\nabla^2\phi = \sigma(t)\,\delta[x - x_0 + f_1(t)]\,\delta[y - y_0 + f_2(t)]\,\delta[z - z_0 + f_3(t)] \tag{11}$$

in the fluid domain R;

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \tag{12}$$

on the free surface $z = 0$;

$$\frac{\partial\phi}{\partial z} = 0 \tag{13}$$

on the bottom of the channel $z = -d$ and

$$\frac{\partial\phi}{\partial y} = 0 \tag{14}$$

on the sides of the channel S_W or $y = \pm b/2$. At $x = \pm\infty$, the potential may be written as

$$\phi = \sum_{m=0}^{\infty}\sum_{n=0} \frac{\cosh[k_m(z+d)]}{\cosh k_m d}\cos\left[\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right]\{A_{mn}^{\pm}\cos[x\sqrt{k_m^2 - (n\pi/b)^2} \mp m\omega t] + B_{mn}^{\pm}\sin[x\sqrt{k_m^2 - (n\pi/b)^2} \mp m\omega t]\} + u_{\pm}x\,, \quad x \to \pm\infty \tag{15}$$

where the upper limit of n is determined by $k_m b/\pi$ and k_m is given in equation (5b). For the fully linearized problem (it has only the term of $m = 1$), it is well known that no wave will propagate to infinity when $\omega < \sqrt{(\pi g/b)}$. In the present case, there are always waves at infinity (those terms with $m > \pi g/(b\omega^2)$, although the waves may be too small to have practical significance if ω is small.

To obtain the relationship between u_+ and u_-, we use

$$\int_0^T \int_S \frac{\partial\phi}{\partial n}\,\mathrm{d}S\,\mathrm{d}t = \int_0^T \int_R \nabla^2\phi\,\mathrm{d}R\,\mathrm{d}t\,.$$

This gives

$$u_+ - u_- = \frac{1}{bd}\int_0^T \sigma(t)\,\mathrm{d}t\,. \tag{16}$$

To obtain the other coefficients in equation (15), the procedure used previously may be followed. If we define

$$\psi_\pm = \cosh[k_m(z+d)]\cos\left[\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right]\sin[x\sqrt{k_m^2-(n\pi/b)^2}\pm m\omega t] \tag{17}$$

and use Green's identity, we have

$$A_{mn}^{\pm} = -\frac{4k_m\cosh k_m d}{Tb\varepsilon_n\sqrt{k_m^2-(n\pi/b)^2}(2k_m d+\sinh 2k_m d)}\int_0^T \sigma(t)\cosh\{k_m[d+z_0-f_3(t)]\}$$
$$\times\cos\left\{\frac{n\pi}{b}\left[\frac{b}{2}+y_0-f_2(t)\right]\right\}\sin\{[x_0-f_1(t)]\sqrt{k_m^2-(n\pi/b)^2}\mp m\omega t]\} \tag{18}$$

where $\varepsilon_0 = 1$ and $\varepsilon_n = 1/2$ if $n > 0$. Similarly if we define

$$\psi_\pm = \cosh[k_m(z+d)]\cos\left[\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right]\cos[x\sqrt{k_m^2-(n\pi/b)^2}\pm m\omega t] \tag{19}$$

we obtain

$$B_{mn}^{\pm} = -\frac{4k_m\cosh k_m d}{Tb\varepsilon_n\sqrt{k_m^2-(n\pi/b^2)}(2k_m d+\sinh 2k_m d)}\int_0^T \sigma(t)\cosh\{k_m[d+z_0-f_3(t)]\}$$
$$\times\cos\left\{\frac{n\pi}{b}\left[\frac{b}{2}+y_0-f_2(t)\right]\right\}\cos\{[x_0-f_1(t)]\sqrt{k_m^2-(n\pi/b)^2}\mp m\omega t]\}\,. \tag{20}$$

4. A three-dimensional source in the open sea

The problem under consideration now satisfies equations (11)–(13). A similar case has been considered by Clement and Ferrant [14]. They obtained the solution of the Green function for a source undergoing heave motion. In the present case, motion is not limited to heave. The expansion at infinity may be written as

$$\phi = \sum_{m=1}^{\infty}\frac{\cosh[k_m(z+d)]}{\cosh k_m d}\frac{1}{\sqrt{k_m r}}[A(\theta)\cos(k_m r - m\omega t) + B(\theta)\sin(k_m r - m\omega t)]\,,\quad r\to\infty \tag{21}$$

where the polar coordinate system (r, θ) is defined as

$$x = r\cos\theta\,,\qquad y = r\sin\theta\,. \tag{22}$$

A distinctive feature of this problem is that the potential tends to zero at infinity. Thus, the corresponding wave elevation itself at $r = \infty$ is of little practical interest. However, the values of $A(\theta)$ and $B(\theta)$ are important. For example, they can be used in the fully linearized wave radiation and wave diffraction problem to calculate the wave damping coefficients and exciting forces.

We define

$$\psi = \cosh k_m(z+d)\sin[k_m r\cos(\theta-\beta) - m\omega t + \pi/4]\,. \tag{23}$$

As before, the Green's identity gives

$$\int_0^T \int_{-d}^0 \int_0^{2\pi} r\left[\psi\frac{\partial\phi}{\partial r} - \phi\frac{\partial\psi}{\partial r}\right]_{r\to\infty} \mathrm{d}\theta\,\mathrm{d}z\,\mathrm{d}t = \int_0^T \sigma(t)\cosh\{k_m[d+z_0-f_3(t)]\} \times \sin[k_m r_0\cos(\theta_0-\beta) - k_m f_1(t)\cos\beta - k_m f_2(t)\sin\beta - m\omega t + \pi/4]\,\mathrm{d}t \tag{24}$$

where r_0 and θ_0 are obtained from equation (22) with x and y being replaced by x_0 and y_0. Using equations (21) and (23), the left-hand side of above equation L becomes

$$\begin{aligned} L = &-\frac{\sqrt{k_m r}}{\cosh k_m d}\int_{-d}^0 \cosh^2 k_m(z+d)\int_0^{2\pi}\int_0^T \\ &\times \langle A_m(\theta)\{\sin[k_m r\cos(\theta-\beta)-m\omega t]\sin(k_m r - m\omega t) \\ &+ \cos(\theta-\beta)\cos[k_m r\cos(\theta-\beta) - m\omega t + \pi/4]\cos(k_m r - m\omega t)\} \\ &- B_m(\theta)\{\sin[k_m r\cos(\theta-\beta)-m\omega t]\cos(k_m r - m\omega t) \\ &- \cos(\theta-\beta)\cos[k_m r\cos(\theta-\beta) - m\omega t + \pi/4]\sin(k_m r - m\omega t)\}\rangle\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}z \\ = &-\frac{\sqrt{k_m r}}{\cosh k_m d}\int_{-d}^0 \cosh^2 k_m(z+d)\int_0^{2\pi} A_m(\theta)\langle\cos\{k_m r[1-\cos(\theta-\beta)] - \pi/4\} \\ &- B_m(\theta)\sin\{k_m r[1-\cos(\theta-\beta)] - \pi/4\}\rangle\,\frac{T}{2}[1+\cos(\theta-\beta)]\,\mathrm{d}\theta\,\mathrm{d}z\,, \quad r\to\infty\,. \end{aligned}$$

The integration with respect to θ may be performed by using the stationary phase method [15]. It gives

$$\int_0^{2\pi} F(\theta)\exp\{ik_m r[1-\cos(\theta-\beta)] - i\pi/4\}[1+\cos(\theta-\beta)]\,\mathrm{d}\theta \to 2F(\beta)\frac{\sqrt{2\pi}}{\sqrt{k_m r}}\,, \quad r\to\infty\,.$$

Thus

$$L = -\sqrt{2\pi}A_m(\beta)T\frac{2k_m d + \sinh 2k_m d}{4k_m\cosh k_m d}\,.$$

Substituting this into equation (24), we obtain

$$A_m(\beta) = -\frac{4k_m\cosh k_m d}{(2k_m d + \sinh 2k_m d)T\sqrt{2\pi}}\int_0^T \sigma(t)\cosh\{k_m[d+z_0-f_3(t)]\} \times \sin[k_m r_0\cos(\theta_0-\beta) - k_m f_1(t)\cos\beta - k_m f_2(t)\sin\beta - m\omega t + \pi/4]\,\mathrm{d}t\,. \tag{25}$$

Similarly if we define

$$\psi = \cosh k_m(z+d)\cos[k_m r\cos(\theta-\beta) - m\omega t + \pi/4] \tag{26}$$

we obtain

$$B_m(\beta) = \frac{4k_m \cosh k_m d}{(2k_m d + \sinh 2k_m d)T\sqrt{2\pi}} \int_0^T \sigma(t) \cosh\{k_m[d + z_0 - f_3(t)]\}$$
$$\times \cos[k_m r_0 \cos(\theta_0 - \beta) - k_m f_1(t) \cos\beta - k_m f_2(t) \sin\beta - m\omega t + \pi/4]\, dt\,. \qquad (27)$$

We now apply the above result to a special case that a source of constant strength moving in a circular path with a constant angular velocity ω in a horizontal plane. We define

$$\sigma(t) = \sigma_0\,, \quad f_1(t) = -a\cos(\omega t + \gamma)\,, \quad f_2(t) = -a\sin(\omega t + \gamma)\,, \quad f_3(t) = x_0 = y_0 = 0\,.$$

The velocity potential or the Green function of this case with $d = \infty$ has been obtained by Havelock [16], which has been used by Wu and Eatock Taylor [17] for solving the problem of a submerged sphere in a circular path. Substituting the above equation into (25) and (27) and using [18]

$$\exp(\pm iz\cos\theta) = \sum_{n=-\infty}^{\infty} J_n(z) \exp[in(\theta \pm \pi/2)] \qquad (28)$$

where $J_m(z)$ are Bessel functions, we obtain

$$A_m(\beta) = -\frac{4\sigma_0 k_m \cosh k_m d \cosh[k_m(d + z_0)]}{(2k_m d + \sinh 2k_m d)T\sqrt{2\pi}} J_m(k_m a) \sin[m(\gamma - \beta + \pi/2) + \pi/4]\,, \qquad (29a)$$

$$B_m(\beta) = \frac{4\sigma_0 k_m \cosh k_m d \cosh[k_m(d + z_0)]}{(2k_m d + \sinh 2k_m d)T\sqrt{2\pi}} J_m(k_m a) \cos[m(\gamma - \beta + \pi/2) + \pi/4]\,. \qquad (29b)$$

Substituting above equations into (21), we have

$$\phi = \sum_{m=1}^{\infty} \frac{4\sigma_0 k_m \cosh k_m d \cosh[k_m(d + z_0)]}{(2k_m d + \sinh 2k_m d)T\sqrt{2\pi}} \frac{1}{\sqrt{k_m r}} J_m(k_m a)$$
$$\times \sin[k_m r + m(\theta - \omega t - \beta) - m\pi/2 - \pi/4]\,. \qquad (30)$$

Let $d = \infty$ and $\sigma = -4\pi$ in this equation. We may compare the result with the asymptotical expansion [17, eq. (29)] of Havelock's solution. It is easy to confirm that apart from different notations having been used these two results are identical.

5. Applications

5.1. Far field equation for drift force

The equations derived above are asymptotical expansions of the potential due to a source at infinity. The results are different from those of the fully linearized theory. In the present case, the wave has an infinite number of components with frequencies $n\omega$ ($n = 1, 2, \ldots$) in contrast to a single component with frequency ω in the linearized problem. Such a difference has important implications to the equations for calculating the forces on a submerged body undergoing large amplitude motions. In particular the far field equation for the drift force has to be modified, as shown below.

The force on a submerged body may be obtained from the following equation [15]

$$\mathbf{F} = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_0} \phi \mathbf{n} \, \mathrm{d}S + \rho \int_{S_0} \left(\frac{\partial \phi}{\partial n} \nabla \phi - \frac{1}{2} \nabla \phi \nabla \phi \mathbf{n} \right) \mathrm{d}S , \tag{31}$$

where S_0 is the body surface and ρ is the density of the fluid. The normal derivative of the potential in this equation is usually known from the following rigid body surface boundary condition

$$\frac{\partial \phi}{\partial n} = \mathbf{V} \cdot \mathbf{n} , \tag{32a}$$

$$\mathbf{V} = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r} , \tag{32b}$$

where $\mathbf{U}$, $\mathbf{\Omega}$ and $\mathbf{r}$ are translational velocity, angular velocity of the body and position vector, respectively. Using Stokes theorem, equation (31) may also be written as

$$\mathbf{F} = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_0} \phi \mathbf{n} \, \mathrm{d}S + \frac{1}{2} \rho \int_{S_0} \left[\frac{\partial \phi}{\partial n} \nabla \phi - \nabla(\phi - \mathbf{V} \cdot \mathbf{r})) \nabla \phi \mathbf{n} \right] \mathrm{d}S .$$

Use is made of the identity [19]

$$\int_{S_0} \nabla \psi \, \nabla \phi n_j \, \mathrm{d}S = \int_{S_0} \frac{\partial^2 \psi}{\partial n \, \partial x_j} \phi \, \mathrm{d}S$$

if $\partial \psi / \partial n = 0$ on S_0, where n_j and x_j $(j = 1, 2, 3)$ are components of $\mathbf{n}$ and $\mathbf{r}$ in x, y and z directions, respectively. The equation for the force becomes

$$\begin{aligned} F_j &= -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_0} \phi n_j \, \mathrm{d}S + \frac{1}{2} \rho \int_{S_0} \left(\frac{\partial \phi}{\partial n} \frac{\partial \phi}{\partial x_j} - \frac{\partial^2 \phi}{\partial n \, \partial x_j} \phi \right) \mathrm{d}S \\ &= -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_0} \phi n_j \, \mathrm{d}S - \frac{1}{2} \rho \int_{S_F + S_\infty} \left(\frac{\partial \phi}{\partial n} \frac{\partial \phi}{\partial x_j} - \frac{\partial^2 \phi}{\partial n \, \partial x_j} \phi \right) \mathrm{d}S . \end{aligned}$$

We now only consider the horizontal forces $(j = 1, 2)$. Substituting the free surface boundary condition on the potential into this equation and integrating both sides with respect t from 0 to T, we obtain

$$\bar{F}_j = -\frac{1}{2T} \rho \int_0^T \mathrm{d}t \int_{S_\infty} \left(\frac{\partial \phi}{\partial n} \frac{\partial \phi}{\partial x_j} - \frac{\partial^2 \phi}{\partial n \, \partial x_j} \phi \right) \mathrm{d}S \tag{33}$$

in which the result has been divided by T. It should be noticed that for problem in the channel this equation is applicable only when $j = 1$.

Similar equation can be derived for the moment about z on a three-dimensional body in the open water. We have [15]

$$M_z = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_0} \phi(n_2 x - n_1 y) \, \mathrm{d}S + \rho \int_{S_0} \left[\frac{\partial \phi}{\partial n} (y \phi_x - x \phi_y) - \frac{1}{2} \nabla \phi \nabla \phi (n_2 x - n_1 y) \right] \mathrm{d}S . \tag{34}$$

Following the above derivation we obtain

$$M_z = -\rho \frac{d}{dt} \int_{S_0} \phi(n_2 x - n_1 y)\, dS + \frac{1}{2}\rho \int_{S_0} \Big[\frac{\partial \phi}{\partial n}(y\phi_x - x\phi_y) - \frac{1}{2}\nabla(\phi - \mathbf{V}\cdot\mathbf{r})\nabla\phi(n_2 x - n_1 y)$$

$$+ \phi(V_2 n_1 + V_1 n_2) \Big] dS$$

$$= -\rho \frac{d}{dt} \int_{S_0} \phi(n_2 x - n_l y)\, dS + \frac{1}{2}\rho \int_{S_0} \left[\frac{\partial \phi}{\partial n}(y\phi_x - x\phi_y) - \frac{\partial(y\phi_x - x\phi_y)}{\partial n}\phi \right] dS$$

$$= -\rho \frac{d}{dt} \int_{S_0} \phi(n_2 x - n_l y)\, dS - \frac{1}{2}\rho \int_{S_F + S_\infty} \left[\frac{\partial \phi}{\partial n}(y\phi_x - x\phi_y) - \frac{\partial(y\phi_x - x\phi_y)}{\partial n}\phi \right] dS$$

where V_j $(j = 1, 2, 3)$ are the components of **V**, in x, y and z directions, respectively. This gives the mean drift moment

$$\bar{M}_z = -\frac{1}{2T}\rho \int_0^T \int_{S_\infty} \left[\frac{\partial \phi}{\partial n}(y\phi_x - x\phi_y) - \frac{\partial(y\phi_x - x\phi_y)}{\partial n}\phi \right] dS\,. \tag{35}$$

The above equations are valid for combined radiation and diffraction problem provided the free surface boundary condition can be linearized. We first restrict our discussion to the case without the incident wave. The asymptotic expansion of the potential at infinity can then be obtained from the results in equations (4), (15) and (21), although the coefficients in these equations are now due to a distribution of the source over the body surface rather than a single source. Substituting these expansions into (33) and (35), we obtain

$$\bar{F}_1 = -\frac{\omega^2}{4g}\rho \sum_{m=1}^{\infty} m^2\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right)(A_m^{+2} + B_m^{+2} - A_m^{-2} - B_m^{-2}) \tag{36}$$

for the two-dimensional problem;

$$\bar{F}_1 = -\frac{b\omega^2}{4g}\rho \sum_{m=1}^{\infty}\sum_{n=0} \frac{m^2[k_m^2 - (n\pi/b)^2]}{k_m^2}\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right)(A_{mn}^{+2} + B_{mn}^{+2} - A_{mn}^{-2} - B_{mn}^{-2}) \tag{37}$$

for the problem in the channel and

$$\bar{F}_1 = -\frac{1}{4}\rho \sum_{m=1}^{\infty} \tanh k_m d\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right) \int_0^{2\pi} \{[A_m(\theta)]^2 + [B_m(\theta)]^2\} \cos\theta\, d\theta \tag{38a}$$

$$\bar{F}_2 = -\frac{1}{4}\rho \sum_{m=1}^{\infty} \tanh k_m d\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right) \int_0^{2\pi} \{[A_m(\theta)]^2 + [B_m(\theta)]^2\} \sin\theta\, d\theta \tag{38b}$$

$$\bar{M}_z = -\frac{1}{4}\rho \sum_{m=1}^{\infty} \frac{\tanh k_m d}{k_m}\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right) \int_0^{2\pi} \left[A_m(\theta)\frac{\partial B_m(\theta)}{\partial \theta} - B_m(\theta)\frac{\partial A_m(\theta)}{\partial \theta} \right] d\theta \tag{38c}$$

in the open sea.

The above results are for the case in which no incoming wave exists. When there are incoming waves, these results have to be modified using a similar derivation. Take the two-dimensional case as an example. If the potential due to the incoming wave can be written as

$$\phi_0 = \sum_{m=0}^{\infty} \frac{\cosh[k_m(z+d)]}{\cosh k_m d} [C_m^+ \cos(k_m x + m\omega t) + D_m^+ \sin(k_m x + m\omega t)$$
$$+ C_m^- \cos(k_m x - m\omega t) + D_m^- \sin(k_m x + m\omega t)] \tag{39}$$

equation (36) becomes

$$\bar{F}_1 = -\frac{\omega^2}{4g}\rho \sum_{m=1}^{\infty} m^2\left(1 + \frac{2k_m d}{\sinh 2k_m d}\right)$$
$$\times (A_m^{+2} + B_m^{+2} - A_m^{-2} - B_m^{-2} + 2A_m^+C_m^+ + 2B_m^+D_m^+ - 2A_m^-C_m^- - 2B_m^-D_m^-)\,. \tag{40}$$

5.2. Results for a submerged ellipsoid in translational motion

To use above equations for a real body needs the solution of source distribution. For a deeply submerged body it may be taken as solution in the infinite fluid domain. We shall consider an ellipsoid as an example.

For a ellipsoid in translational oscillation in an unbounded fluid, the velocity potential can be expanded as

$$\phi = U\phi_1 + V\phi_2 + W\phi_3 \tag{41}$$

where U, V and W are velocities in x, y and z directions, respectively. The velocity potential components may be written in the form [20]

$$\phi_j = -\frac{2}{2-\alpha_j}\frac{\partial \psi_j}{\partial x_j} \tag{42}$$

where

$$\psi_j = \frac{1}{4\pi}\int\int\int \frac{\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}} \tag{43a}$$

and

$$\alpha_j = a_1 a_2 a_3 \int_0^{\infty} \frac{\mathrm{d}\lambda}{(a_j^2+\lambda)[(a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda)]^{1/2}} \tag{43b}$$

with a_j $(j = 1, 2, 3)$ being the semi-lengths of the three principal axes. The integration in equation (43a) is over the volume of the ellipsoid. Since it is in oscillation, we may write

$$\xi = x_0 - f_1(t)\,, \qquad \eta = y_0 - f_2(t)\,, \qquad \zeta = z_0 - f_3(t)\,. \tag{44}$$

Equation (43a) becomes

$$\psi_j = \frac{1}{4\pi}\int\int\int \frac{\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0}{[(x-x_0+f_1)^2 + (y-y_0+f_2)^2 + (z-z_0+f_3)^2]^{1/2}}\,. \tag{45}$$

It is now evident that equation (42) can be written as

$$\phi_j = \frac{2}{2-\alpha_j}\frac{1}{4\pi}\int\int\int \frac{\partial}{\partial x_{0j}}$$

$$\times \left(\frac{1}{[(x-x_0+f_1)^2+(y-y_0+f_2)^2+(z-z_0+f_3)^2]^{1/2}}\right) dx_0\, dy_0\, dz_0 \,. \tag{46}$$

This physical meaning of this equation is that the potential is represented by a distribution of dipoles with constant strength over the volume of the ellipsoid. Thus, using equations (41), (25) and (27), we obtain in deep water ($d=\infty$)

$$A_m(\beta) = -\frac{2}{2-\alpha_j}\frac{2k_m^2}{T\sqrt{2\pi}}\int\int\int dx_0\, dy_0\, dz_0 \int_0^T \exp[k_m(z_0-f_3)]$$
$$\times \{U\cos\beta\cos[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]$$
$$+V\sin\beta\cos[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]$$
$$+W\sin[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]\}\, dt\,,$$

$$B_m(\beta) = \frac{2}{2-\alpha_j}\frac{2k_m^2}{T\sqrt{2\pi}}\int\int\int dx_0\, dy_0\, dz_0 \int_0^T \exp[k_m(z_0-f_3)]$$
$$\times \{-U\cos\beta\sin[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]$$
$$-V\sin\beta\sin[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]$$
$$+W\cos[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]\}\, dt\,.$$

Using

$$U=-df_1/dt\,,\qquad V=-df_2/dt\,,\qquad W=-df_3/dt\,,$$

the above equation may also be written as

$$A_m(\beta) = -\frac{2}{2-\alpha_j}\frac{2k_m}{T\sqrt{2\pi}}\int\int\int dx_0\, dy_0\, dz_0 \int_0^T \left\langle\frac{\partial}{\partial t}\{\exp[k_m(z_0-f_3)]\right.$$
$$\times\sin[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]\}+m\omega$$
$$\times\exp[k_m(z_0-f_3)]\cos[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t$$
$$\left.+\pi/4]\right\rangle dt$$
$$= -\frac{2}{2-\alpha_j}\frac{2k_m m\omega}{T\sqrt{2\pi}}\int\int\int dx_0\, dy_0\, dz_0 \int_0^T \exp[k_m(z_0-f_3)]$$
$$\times\cos[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]\, dt\,,$$

$$B_m(\beta) = \frac{2}{2-\alpha_j}\frac{2k_m m\omega}{T\sqrt{2\pi}}\int\int\int dx_0\, dy_0\, dz_0 \int_0^T \exp[k_m(z_0-f_3)]$$
$$\times\sin[k_m r_0\cos(\theta_0-\beta)-k_m f_1\cos\beta-k_m f_2\sin\beta-m\omega t+\pi/4]\, dt\,.$$

Further from the result in the appendix, we have

$$A_m(\beta) = \frac{Q_m}{T} \int_0^T \exp(-k_m f_3) \cos[k_m f_1 \cos\beta + k_m f_2 \sin\beta + m\omega t - \pi/4]\, dt\,, \tag{47a}$$

$$B_m(\beta) = -\frac{Q_m}{T} \int_0^T \exp(-k_m f_3) \sin[k_m f_1 \cos\beta + k_m f_2 \sin\beta + m\omega t - \pi/4]\, dt\,, \tag{47b}$$

where

$$Q_m = -\frac{2}{2-\alpha_j} 4\pi a_1 a_2 m\omega \exp(-k_m h) \times \sum_{n=0}^{\infty} \frac{1}{2^n n!} \frac{(k_m a_3)^{2n+1}}{[k_m\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta}]^{n+3/2}} J_{n+3/2}(k_m\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})\,. \tag{48}$$

Let

$$f_1 = f\cos\gamma\,, \qquad f_2 = f\sin\gamma\,.$$

Equations (47) may be written as

$$A_m(\beta) - iB_m(\beta) = \frac{Q_m}{T}\int_0^T \exp[-k_m f_3 + ik_m f\cos(\beta-\gamma) + im\omega t - i\pi/4]\, dt\,.$$

Because of equations (28), this equation becomes

$$A_m(\beta) - iB_m(\beta) = Q_m \exp(-i\pi/4) \sum_{p=-\infty}^{\infty} i^p \exp(ip\beta) I(p,m) \tag{49}$$

where

$$I(p,m) = \frac{1}{T}\int_0^T \exp(-k_m f_3 - ip\gamma + im\omega t) J_p(k_m f)\, dt\,. \tag{50}$$

The above equations can be substituted into (38) to calculate the drift forces on a submerged ellipsoid in translation. There is no restriction on the dimension of the ellipsoid, provided its disturbance on the free surface is small. When it is slender, we need retain only the term of $n = 0$ in equation (48) as adopted by Havelock [20] for a different problem.

When $a_1 = a_2 = a_3 = a$, which represents a sphere, we may use [21]

$$\frac{(z/2)^\mu}{\Gamma(\mu+1)} = \sum_{n=0}^{\infty} \frac{(z/2)^n}{n!} J_{n+\mu}(z)\,.$$

Equation (48) becomes

$$Q_m = -2\sqrt{2\pi} a^3 m\omega k_m a \exp(-k_m h)\,. \tag{51}$$

Substituting these equations into (38), we obtain

$$\bar{F}_1 - i\bar{F}_2 = -\frac{\pi i}{2}\rho \sum_{m=1}^{\infty} Q_m^2 \sum_{p=-\infty}^{\infty} I(p,m)I^*(p-1,m)\,, \tag{52a}$$

$$\bar{M}_z = -\frac{\pi}{2}\rho \sum_{m=1}^{\infty} \frac{Q_m^2}{k_m} \sum_{p=-\infty}^{\infty} pI^*(p,m)I(p,m)\,. \tag{52b}$$

When the sphere is in a circular motion in a horizontal plane, as defined by the following equation

$$f_3 = 0\,, \qquad f = \text{constant}\,, \qquad \gamma = \omega t\,,$$

equation (50) can be simplified as

$$I(p,m) = \begin{cases} J_m(k_m f) & p = m \\ 0 & p \neq m \end{cases} \tag{53}$$

This leads to

$$\bar{F}_1 = \bar{F}_2 = 0\,, \tag{54a}$$

$$\bar{M}_z = -\frac{\pi}{2}\rho \sum_{m=1}^{\infty} \frac{Q_m^2}{k_m} m J_m^2(k_m f)$$
$$= -4\pi^2\omega^4 a^6/g \sum_{m=1}^{\infty} m^5 \exp(-2k_m h) J_m^2(k_m f)\,. \tag{54b}$$

For a sphere moving in a circular path, the tangential and radial forces do not change with time in the coordinate system fixed on the body, when the steady state has been reached. The mean moment about the centre of the circle should be equal to the tangential force multiplied by the radius of the circle. This enables us to compare equation (54b) with Rf, where R is the tangential force given in equation (12) of [16] (notice different notations have been used here). The results can be seen identical.

6. Conclusions

A common feature in the derivation of various identities in the linearized and second-order potential flow problems has been noticed. The method is then applied to calculate the wave radiation by a submerged source, which does not require the full solution of problem. The definition of ψ plays the crucial role. It was chosen as another radiation or diffraction potential in the linearized problem so that some identities were established between one radiation (or diffraction) and another. We notice that in this paper ψ is in fact an incident potentials. When the Green's identity is used, the integration of the potentials over the surface will cancel each other apart from where the radiation condition is not satisfied by ψ (in the two-dimensional case) or it is a stationary point of the integrand (in the three-dimensional case). Evidently one can define an appropriate ψ to obtain other results. However, in some cases it may be more difficult to obtain ψ than to solve the original problem. The conclusion therefore is that the method may be applicable only if ψ can be chosen as a known function or that having been previously obtained.

The results obtained in this paper for an isolated source are useful to a real body if its source distribution can be easily found. Thus, they are particularly effective for problems related to slender bodies or deeply submerged bodies undergoing large amplitude motion, as demonstrated by the example in the paper.

Appendix

To calculate the integral over the volume of the ellipsoid before equations (47), we consider the following integral

$$I = \int\int\int \exp[k_m z + ik_m(x\cos\beta + y\sin\beta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_{-a_1}^{a_1}\int_{-a_2\sqrt{1-x^2/a_1^2}}^{a_2\sqrt{1-x^2/a_1^2}}\int_{-h-a_3\sqrt{1-x^2/a_1^2-y^2/a_2^2}}^{-h+a_3\sqrt{1-x^2/a_1^2-y^2/a_2^2}} \exp[k_m z + ik_m(x\cos\beta + y\sin\beta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \frac{2}{k_m}\exp(-k_m h)\int_{-a_1}^{a_1}\int_{-a_2\sqrt{1-x^2/a_1^2}}^{a_2\sqrt{1-x^2/a_1^2}} \sinh k_m a_3\sqrt{1-x^2/a_1^2-y^2/a_2^2}\exp[ik_m(x\cos\beta$$
$$+ y\sin\beta)\,\mathrm{d}x\,\mathrm{d}y$$

Let

$$x = a_1 r\cos\theta \qquad y = a_2 r\sin\theta$$

The above equation becomes

$$I = \frac{2a_1a_2}{k_m}\exp(-k_m h)\int_0^1\int_0^{2\pi} \sinh[k_m a_3\sqrt{1-r^2}]\exp[ik_m r\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta}\cos\theta]r\,\mathrm{d}\theta\,\mathrm{d}r$$
$$= \frac{4\pi a_1a_2}{k_m}\exp(-k_m h)\int_0^1 \sinh[k_m a_3\sqrt{1-r^2}]J_0(k_m r\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})r\,\mathrm{d}r$$

Replacing r with $\sin\theta$, we obtain

$$= \frac{4\pi a_1a_2}{k_m}\exp(-k_m h)\int_0^{\pi/2} \sinh(k_m a_3\cos\theta)J_0(k_m\sin\theta\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})\sin\theta\cos\theta\,\mathrm{d}\theta$$
$$= \frac{4\pi a_1a_2}{k_m}\exp(-k_m h)\int_0^{\pi/2} J_0(k_m\sin\theta\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})$$
$$\times\sum_{n=0}^{\infty}\frac{(k_m a_3)^{2n+1}}{(2n+1)!}\sin\theta\cos^{2n+2}\theta\,\mathrm{d}\theta$$
$$= \frac{2\pi a_1a_2}{k_m}\exp(-k_m h)\sqrt{2\pi}\sum_{n=0}^{\infty}\frac{1}{2^n n!}\frac{(k_m a_3)^{2n+1}}{[k_m\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})]^{n+3/2}}$$
$$\times J_{n+3/2}(k_m\sqrt{a_1^2\cos^2\beta + a_2^2\sin^2\beta})$$

where equation (11.4.10) of [18] has been used.

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